

MICROSCOPIC THEORY OF THE CASIMIR EFFECT

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Based on the photon-exciton Hamiltonian a microscopic theory of the Casimir problem for dielectrics is developed. Using well-known many-body techniques we derive a perturbation expansion for the energy which is free from divergences. In the continuum limit we turn off the interaction at a distance smaller than a cut-off distance a to keep the energy finite. We will show that the macroscopic theory of the Casimir effect with hard boundary conditions is not well defined because it ignores the finite distance between the atoms, hence is including infinite self-energy contributions. Nevertheless for disconnected bodies the latter do not contribute to the force between the bodies. The Lorentz-Lorenz relation for the dielectric constant that enters the force is deduced in our microscopic theory without further assumptions.

The photon Green's function can be calculated from a Dyson type integral equation. The geometry of the problem only enters in this equation through the region of integration which is equal to the region occupied by the dielectric. The integral equation can be solved exactly for various plain and spherical geometries without using boundary conditions. This clearly shows that the Casimir force for dielectrics is due to the forces between the atoms.

Convergence of the perturbation expansion and the metallic limit are discussed. We conclude that for any dielectric function the transverse electric (TE) mode does not contribute to the zero-frequency term of the Casimir force.

I. INTRODUCTION

More than 50 years ago Casimir published two completely different papers [1] [2] on what was later called the Casimir effect: One [1] was a cumbersome microscopic theory, the other [2] a very elegant macroscopic one. From the latter one may get the impression that the problem is rather simple, namely, quantization of the free electromagnetic field in a region with boundaries. However, the results of such macroscopic calculations (the so-called mode summation method) are notoriously divergent ([3] and the many references therein). Although by suitable regularizations one may get reasonable results, the ultraviolet divergences cannot be eliminated in general. This shows that the macroscopic problem (with hard boundary conditions) is not well-posed. A second weak point of the macroscopic method is that it completely ignores the fact that the real Casimir force is the result of electromagnetic interaction between neutral atoms and not a free problem. This was already the point of view of Casimir and Polder in their first paper [1]. It is clear from the above considerations that a real understanding of the Casimir problem can only come from a microscopic theory.

In the following we develop such a theory for dielectrics and use it to solve various Casimir problems explicitly. The basis is the photon-exciton Hamiltonian given in the next section. The interaction consists of the non-relativistic dipole coupling to the radiation field and the dipole-dipole approximation of the Coulomb interaction. In Section III we derive the perturbation expansion for the energy E for arbitrarily spaced atoms. This energy is finite since infinite self-energy contributions of the dipoles are excluded.

When taking the limit to a continuous dipole distribution one has to be careful because of the dipole-dipole approximation. We turn off the interaction at an average distance a between the dipoles. This was also done by Marachevsky [6], leading to a finite energy for a dilute ball. To extract the classical Casimir energy from E we have to introduce a further ultraviolet cut-off $e^{-\delta k}$. E^δ splits then into two parts: $E^\delta = E_D^\delta - E_G^\delta$. In the limit $\delta \rightarrow 0$ only E^δ remains finite, whereas E_D^δ and E_G^δ are divergent. In the limit $a \rightarrow 0$ also E is divergent. E_D^δ is for $\delta, a \rightarrow 0$ formally equivalent to the usual Casimir energy E_D as given by Candelas [8]. It is always infinite.

To calculate the force between separated bodies in regions A and B , one has to use the free energy F . It splits into three finite parts: $F = F_A + F_B + F_{AB}$. F_A and F_B contain the free energies of the two isolated bodies. They are finite since the finite distance between the atoms is taken into account and self-energies are excluded. Only the interaction energy F_{AB} depends on the distance of the bodies and contributes to the force. In the limit $a \rightarrow 0$ only

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this energy remains finite. For the polarization we obtain the Lorentz-Lorenz relation without further assumptions. The force between disconnected bodies can be calculated by differentiating F with respect to the distance or by a contour integral over a region containing one body. Only the stress field outside the bodies contributes to the force and is finite also when the cut-off distance goes to 0.

In Section IV the Green's function of the total electromagnetic field for a continuous dipole distribution field is discussed. It is given by a Dyson type integral equation. In this equation the many-body problem is separated from the geometry. The latter only enters through the region of integration which agrees with the region of space occupied by the dielectric. We identify a class of geometries which allow exact solution of the integral equation. As applications we consider in Section V and VI plain geometries and we re-derive the Lifshitz formula for the Casimir force between two plates [7]. In Section VII we solve the integral equation for the Green's function of the electromagnetic field for a dielectric ball.

In Section VIII the metallic limit $\varepsilon \rightarrow \infty$ is considered. Within our theory because of the Lorentz-Lorenz relation this means to put the atomic polarization $\alpha_0(u) = 3$ for $u = 0$. It is not clear whether that the perturbation expansion converges. We'll show convergence in flat geometries just for $u = 0$ in the TM mode and for small frequencies $u > 0$ in the TE mode. The TE mode does not contribute in the zero-frequency case $u = 0$ to the Casimir force independent of the model adopted for the dielectric function. This result cannot be deduced from the Lifshitz formula for the plasma model where $\lim_{u \rightarrow 0} \varepsilon(iu)u^2 \neq 0$. The zero-frequency behavior plays an important role in the context of the temperature dependence of the Casimir force and is widely discussed in recent papers [11–17].

II. THE MODEL

We consider N atoms in interaction with the quantized radiation field. The dynamical degrees of freedom of the atoms are their internal excitations, therefore, our system is described by the following photon - exciton Hamiltonian:

$$H = H_0^a + H_0^{ph} + H_1, \quad (1)$$

$$H_0^a = \sum_{\vec{x}, n} E_n b_n^+(\vec{x}) b_n(\vec{x}), \quad (2)$$

$$\begin{aligned} H_0^{ph} &= \frac{1}{2} \int d^3x \left(: \vec{E}(\vec{x})^2 : + : \vec{B}(\vec{x})^2 : \right) \\ &= \sum_{\lambda=1,2} \int d^3x k a_\lambda^+(\vec{k}) a_\lambda(\vec{k}) \end{aligned} \quad (3)$$

$$\begin{aligned} H_1 &= \sum_{\vec{x}} \left[-\frac{e}{mc} \vec{p}_{\vec{x}} \vec{A}(\vec{x}) + \frac{e^2}{2mc^2} : \vec{A}(\vec{x})^2 : \right] \\ &\quad - \frac{1}{2} \sum_{\vec{x} \neq \vec{x}'} \Phi(\vec{x} - \vec{x}'). \end{aligned} \quad (4)$$

The atoms sit at places \vec{x}_j , but for simplicity of the notation we omit the index j and write the sum over all atoms as $\sum_{\vec{x}}$. $b_n^+(\vec{x})$ and $b_n(\vec{x})$ are emission and absorption operators for an atom in state n at the place \vec{x} . The interaction H_1 consists of the usual dipole coupling of non-relativistic radiation theory plus the dipole - dipole interaction Φ . We assume that the distance between the atoms is big compared with the extension of the atoms (Bohr radius). Then Φ is given by

$$\Phi(\vec{x} - \vec{x}') = \sum_{i,j=1}^3 q_i(\vec{x}) q_j(\vec{x}') \Phi_{i,j}(\vec{x} - \vec{x}') \quad (5)$$

$$\Phi_{i,j}(\vec{R}) = \frac{1}{4\pi R^3} \left(\delta_{ij} - 3 \frac{R_i R_j}{R^2} \right), \quad (6)$$

where $\vec{q}(\vec{x})$ denotes the dipole operator of an atom at place \vec{x} . It can be expressed by one-particle matrix elements of the position operator, similarly to the momentum operator $\vec{p}(\vec{x})$:

$$\vec{p}(\vec{x}) = \sum_n (\varphi_0, \vec{p}\varphi_n) b_n(\vec{x}) + (\varphi_n, \vec{p}\varphi_0) b_n^+(\vec{x}). \quad (7)$$

The commutation relations for the emission and absorption operators are

$$[b_m(\vec{x}), b_n^+(\vec{x}')] = \delta_{mn} \delta_{\vec{x}\vec{x}'}, \quad (8)$$

and zero otherwise. $\delta_{\vec{x}\vec{x}'}$ is actually a Kronecker delta, it is $=1$ if $x' = x$ and 0 otherwise. The quantized electromagnetic field is given in the radiation gauge by

$$\vec{A}(\vec{x}) = \frac{1}{(2\pi)^{3/2}} \sum_{\lambda=1,2} \int \frac{d^3 k}{\sqrt{2k}} \vec{\xi}_\lambda(\vec{k}) \left(a_\lambda(\vec{k}) e^{i\vec{k}\vec{x}} + a_\lambda^+(\vec{k}) e^{-i\vec{k}\vec{x}} \right), \quad (9)$$

where $\vec{\xi}_\lambda(\vec{k})$ are transversal polarization vectors:

$$(\vec{\xi}_\lambda(\vec{k}), \vec{\xi}_{\lambda'}(\vec{k})) = \delta_{\lambda\lambda'}, \quad (10)$$

$$(\vec{\xi}_\lambda(\vec{k}), \vec{k}) = 0. \quad (11)$$

The photon emission and absorption operators satisfy

$$[a_\lambda(\vec{k}), a_{\lambda'}^+(\vec{k}')] = \delta_{\lambda\lambda'} \delta(\vec{k} - \vec{k}') \quad (12)$$

and zero otherwise.

III. ENERGY AND THE STRESS TENSOR

We use the Matsubara formalism to calculate the expectation value for the energy for finite temperature. The expectation value of Hamiltonian is given by:

$$E = \langle H \rangle = \frac{\text{Tr}(e^{-\beta H} H)}{\text{Tr}(e^{-\beta H})} = -\partial_\beta \ln \text{Tr}(e^{-\beta H}) \quad (13)$$

With

$$S = e^{-\beta H_0} e^{-\beta H} \quad (14)$$

we obtain

$$\begin{aligned} E &= -\partial_\beta \ln \frac{\text{Tr}(e^{-\beta H_0} S)}{\text{Tr}(e^{-\beta H_0})} - \partial_\beta \ln \text{Tr}(e^{-\beta H_0}) \\ &= -\partial_\beta \ln \langle S \rangle_0 + \langle H_0 \rangle_0. \end{aligned} \quad (15)$$

The term $\langle H_0 \rangle_0$ is just giving the ground state energy of the free system. It is the sum of the excitation energies of the atoms and the black-body radiation energy. The later is 0 for $T \rightarrow 0$, since we use the normal ordering in the definition of the free photon Hamiltonian (3). The normal ordering is necessary for the Hamiltonian to be well-defined. The $\langle H_0 \rangle_0$ -term will be ignored in the following. The expectation value of the S -matrix is given by perturbation theory.

$$\langle S \rangle_0 = 1 - \sum_{n=1}^{\infty} (-1)^n \int_0^\beta d\tau_1 \dots d\tau_n \langle T_\tau H_1(\tau_1) \dots H_1(\tau_n) \rangle_0 \quad (16)$$

where $H_1(\tau)$ is the Heisenberg representation for the interaction Hamiltonian H_1 with imaginary time $\tau = -it$. It is well known that in $\ln \langle S \rangle_0$ only connected graphs contribute [4, 5]. We have

$$\begin{aligned} \ln \langle S \rangle_0 &= \sum_{m=-\infty}^{\infty} \sum_{n=2}^{\infty} \frac{1}{2n} \sum_{\vec{x}_1 \dots \vec{x}_n} \hat{a}_0(i\omega_m, \vec{x}_1) \hat{D}_{i_1 i_2}^0(i\omega_m, \vec{x}_1 - \vec{x}_2) \\ &\quad \dots \hat{a}_0(i\omega_m, \vec{x}_n) \hat{D}_{i_n i_1}^0(i\omega_m, \vec{x}_n - \vec{x}_1) \end{aligned} \quad (17)$$

where the last sum goes over the positions \vec{x}_i of the dipoles with $\vec{x}_1 \neq \vec{x}_2 \dots$ for not including the divergent self-energy contributions. α_0 is the atomic polarization given by

$$\alpha_{ij}(\tau, \vec{x} - \vec{x}') = -\langle T_\tau q_i(\tau, \vec{x}) q_j(0, \vec{x}') \rangle_0 = \alpha_0(\tau, \vec{x}) \delta_{ij} \delta_{xx'} \quad (18)$$

for isotropic atoms. Since α_0 obeys the KMS periodicity $\alpha_0(\tau) = \alpha_0(\tau + \beta)$, it can be expand into a Fourier series

$$\alpha_0(\tau, \vec{x}) = \frac{1}{\beta} \sum_{m=-\infty}^{\infty} \hat{\alpha}_0(i\omega_m, \vec{x}) e^{-i\omega_m \tau}, \quad (19)$$

where

$$\omega_m = \frac{2\pi m}{\beta} \quad (20)$$

are the Matsubara frequencies. The Fourier coefficients in (19) are given by

$$\hat{\alpha}_0(w, \vec{x}) = \int_0^\beta d\tau \alpha_0(\tau, \vec{x}) e^{w\tau}, \quad (21)$$

using the short-hand notation

$$w = i\omega_m = i \frac{2\pi m}{\beta}. \quad (22)$$

The Green's function $\hat{D}_{ij}^0(\omega, \vec{x} - \vec{y})$ in (17) is the Fourier transformed free Green's function of the total electric field given by

$$D_{ij}^0(\tau, \vec{x} - \vec{y}) = \partial_\tau^2 \langle T_\tau A_i(\tau, \vec{x}) A_j(0, \vec{y}) \rangle_0 + \Phi_{ij}(\vec{x} - \vec{y}) \quad (23)$$

In momentum space we get

$$\begin{aligned} \hat{D}_{ij}^0(\omega, \vec{k}) &= \frac{\omega^2 (\delta_{ij} - \frac{k_i k_j}{k^2})}{k^2 - \omega^2} - \frac{k_i k_j}{k^2} = \frac{\omega^2 \delta_{ij} - k_i k_j}{k^2 - \omega^2} \\ &= -\delta_{ij} + \frac{k^2}{k^2 - \omega^2} \sum_{\lambda=1,2} \xi_i^\lambda(\vec{k}) \xi_j^\lambda(\vec{k}) \end{aligned} \quad (24)$$

We get the energy in (15) by differentiating (17) with respect to β and using $\partial_\beta \omega_m = -\omega_m/\beta$. Hence

$$\begin{aligned} E &= \frac{1}{\beta} \sum_{m=-\infty}^{\infty} \omega_m \partial_{\omega_m} \sum_{n=2}^{\infty} \frac{1}{2n} \sum_{\vec{x}_1 \dots \vec{x}_n} \hat{\alpha}_0(i\omega_m, \vec{x}_1) \hat{D}_{i_1 i_2}^0(i\omega_m, \vec{x}_1 - \vec{x}_2) \dots \\ &\quad \dots \hat{\alpha}_0(i\omega_m, \vec{x}_n) \hat{D}_{i_n i_1}^0(i\omega_m, \vec{x}_n - \vec{x}_1). \end{aligned} \quad (25)$$

To calculate the Casimir force with finite temperature we need the free energy given by

$$\begin{aligned} F &= -\frac{1}{\beta} \ln \langle S \rangle_0 \\ &= -\frac{1}{\beta} \sum_{m=-\infty}^{\infty} \sum_{n=2}^{\infty} \frac{1}{2n} \sum_{\vec{x}_1 \dots \vec{x}_n} \hat{\alpha}_0(i\omega_m, \vec{x}_1) \hat{D}_{i_1 i_2}^0(i\omega_m, \vec{x}_1 - \vec{x}_2) \dots \\ &\quad \dots \hat{\alpha}_0(i\omega_m, \vec{x}_n) \hat{D}_{i_n i_1}^0(i\omega_m, \vec{x}_n - \vec{x}_1). \end{aligned} \quad (26)$$

In the limit $T \rightarrow 0$ ($\beta \rightarrow \infty$) the two energies E and F are the same. Indeed the summation over the frequencies ω_m is replaced by an integral:

$$\frac{1}{\beta} \sum_{m=-\infty}^{\infty} f(\omega_m) \longrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} du f(u) \quad \text{for } T \rightarrow 0, \quad (27)$$

and E then transforms into F by partial integration.

It is convenient to put the position dependence of the dipoles in the atomic polarization function: $\alpha_0(\vec{x}) = \sum_{\vec{R}} \alpha_0 \delta(\vec{x} - \vec{R})$ where \vec{R} runs over all positions of the dipoles. For not including self-energy contributions, we have to replace D^0 by D'_0 :

$$D'_0(iw_n, \vec{x}) = \begin{cases} D^0(iw_n, \vec{x}) & \text{for } \vec{x} \neq \vec{0} \\ 0 & \text{for } \vec{x} = \vec{0} \end{cases}. \quad (28)$$

The integration over x can be considered as matrix multiplications. Then the spectral densities of the energies (25,26) can be written as

$$E(\omega) = \frac{1}{2} w \partial_w \sum_{n=2}^{\infty} \frac{1}{n} \text{Tr}((\alpha_0 D'_0)^n) \quad (29)$$

$$F(w) = -\frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n} \text{Tr}((\alpha_0 D'_0)^n) \quad (30)$$

The two energy densities are finite, since self-energy contributions ($\vec{x}_i = \vec{x}_{i+1}$) are excluded by (28).

Next we consider the limit to a continuous distribution of the dipoles. The sums over the positions are then replaced by integrals over the whole space. The polarization function is now

$$\alpha_0(iw_n, \vec{x}) = \alpha_0(iw_n) \theta_K(\vec{x}), \quad (31)$$

where $\theta_K(\vec{x}) = 1$ for \vec{x} in the region K occupied by the dielectric and 0 otherwise. The polarization $\alpha_0(iw_n)$ is replaced by its density. D'_0 is an integral operator with kernel $D'_0(i\omega_n, \vec{x} - \vec{y})$. The limit only exists if D'_0 in (29, 30) contains an ultraviolet cut-off in momentum space or a cut-off at small distances in x -space. This can be seen as follows. According to (29) $(\alpha_0 D'_0)^2$ must be a trace-class operator. This is true if $\alpha_0 D'_0$ is a Hilbert - Schmidt operator, that means its integral kernel must be square integrable

$$\int d^3y \int d^3x |\alpha_0(\vec{x}) D'_0(\vec{x} - \vec{y})|^2 = \|\alpha_0 D'_0\|_{\text{HS}}^2 = \alpha_0^2 \cdot \text{vol}(K) \|D'_0\|_2^2 < \infty. \quad (32)$$

Note that the square of a Hilbert - Schmidt operator is trace-class, and the same is true for all higher powers in (29, 30). The dielectric must have a finite volume and the Green's function be square integrable. But a glance to (24) shows that an ultraviolet cut-off is necessary for this. The problem comes from the dipole - dipole interaction.

In x -space the dipole - dipole term $\Phi_{ij}(\vec{x})$ is not square integrable at $\vec{x} = \vec{0}$. Therefore, we introduce a spatial cut-off at a minimum average distance a between the atoms as it was also done by Marachevsky [6]. We correct the Green's function D^0 for small distances as follows

$$\hat{D}'_0(\omega, \vec{x}) = \hat{D}^0(\omega, \vec{x}) \theta(x - a) = \hat{D}^0(\omega, \vec{x}) - \hat{G}^0(\omega, \vec{x}), \quad (33)$$

where

$$\hat{G}^0(\omega, \vec{x}) = \hat{D}^0(\omega, \vec{x}) \theta(a - x). \quad (34)$$

We show next, that the splitting of $D'_0 = D^0 - G^0$ leads to corresponding splitting in the energy $E = E_D - E_G$. In the sums (29,30) we introduce the first order term in α_0 , which is 0 since $\alpha_0 D'_0(\vec{0}) = 0$, and get

$$\begin{aligned} -\sum_{n=1}^{\infty} \frac{1}{n} \text{Tr}((\alpha_0 D'_0)^n) &= \text{Tr}(\ln(1 - \alpha_0 D^0 + \alpha_0 G^0)) \\ &= \text{Tr}\left(\ln((1 + \alpha_0 G^0)(1 - (1 + \alpha_0 G^0)^{-1} \alpha_0 D^0))\right) \\ &= \text{Tr}(\ln(1 + \alpha_0 G^0) + \ln(1 - \alpha D^0)), \end{aligned} \quad (35)$$

where α is the macroscopic polarizability given by

$$\alpha = (1 + \alpha_0 G^0)^{-1} \alpha_0. \quad (36)$$

If we express α_0 by α in the first term of (35) we get $1 + \alpha^0 G^0 = (1 - \alpha G^0)^{-1}$. We would like to split the trace in (35) into two separate traces. As discussed above this is only possible with an ultraviolet cut-off $e^{-\delta k}$ in the Green's functions. For the inner energy E we then get

$$E = \lim_{\delta \rightarrow 0} (E_D^\delta - E_G^\delta) \quad (37)$$

with

$$E_D^\delta(w) = w \partial_w \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} ((\alpha D_\delta^0)^n) \quad (38)$$

and

$$E_G^\delta(w) = w \partial_w \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} ((\alpha G_\delta^0)^n) \quad (39)$$

and similarly for the free energy F . In the limit $\delta \rightarrow 0$ E remains finite but E_D^δ and E_G^δ are both divergent. In the limit $a \rightarrow 0$ also E is divergent. The energy E_D^δ has the similar form as in (29), only that the atomic polarizability is replaced by the macroscopic polarizability α defined in (36) and D'_0 is replaced by the cut-off Green's function D_δ^0 . As the cut-off parameters a and δ go to 0, we have

$$G_\delta^0(\omega, \vec{x}) \rightarrow -\frac{1}{3} \delta(\vec{x}) \quad (40)$$

and we get from (36)

$$\alpha(\omega, \vec{x}, \vec{x}') = \frac{\alpha_0(\omega, \vec{x})}{1 - \frac{1}{3}\alpha_0(\omega, \vec{x})} \delta(\vec{x} - \vec{x}') \quad (41)$$

the Lorentz-Lorenz relation for the macroscopic polarizability. We shall use this relation in later calculations.

E_D^δ is for $\delta, a \rightarrow 0$ formally equivalent to the energy E_D that is usually calculated in the Casimir problem. To bring the energy E_D^δ in a convenient form, we perform the derivative in (38) using the relation

$$w \partial_w D_\delta^0 = 2 (D_\delta^0 + (D_\delta^0)^2). \quad (42)$$

(38) then becomes

$$E_D^\delta(w) = \frac{1}{2} \text{Tr} \sum_n' ((w \partial_w \alpha) + 2(1 + \alpha)) D_\delta^0 (\alpha D_\delta^0)^n, \quad (43)$$

where the prime in the summation means that we just sum up in orders of $\alpha > 0$. The last part $\sum_n D^0 (\alpha D^0)^n$ is the Green's function of the total electromagnetic field

$$\begin{aligned} \langle E_i(\vec{x}), E_j(\vec{x}') \rangle &= D^0(\vec{x} - \vec{x}') \\ &+ \sum_{n=1}^{\infty} \int dy_1 \dots dy_n D_{ik_1}^0(\vec{x} - \vec{y}_1) \alpha(y_1) \dots \alpha(y_n) D_{k_n j}^0(\vec{y}_n - \vec{x}'), \end{aligned} \quad (44)$$

where α is given by (41). One can get this Green's function also by ordinary perturbation theory and Dyson resummation. It will be discussed extensively in the next section. Note that in (44) no cut-off is needed.

With $\vec{B} = i/w \vec{\nabla} \wedge \vec{E}$ and the relation $\langle \varepsilon \vec{E}(\vec{x})^2 \rangle = \langle \vec{B}(\vec{x})^2 \rangle + \text{div}$ we finally get for energy density

$$e_D(w, \vec{x}) = \frac{1}{2} \left(\langle (\partial_w w \varepsilon(w, \vec{x})) E_i(\vec{x}) E_i(\vec{x}) \rangle' + \langle B_i(\vec{x}) B_i(\vec{x}) \rangle' \right) \quad (45)$$

where $\varepsilon = 1 + \alpha$ and the prime means, that the 0-th order in α is suppressed. This corresponds only to the T_{00} -component of electromagnetic stress tensor if there is no dispersion. The energy density in (45) was also used by Candelas [8] (cited by [9]). However this energy density is only well defined outside the dielectric. The total energy is infinite.

For two disconnected dielectrics it is possible to express the energy F by three parts: two containing just the energies of the isolated dielectrics and one containing the interaction energies of the two bodies. Let $\alpha_0(\vec{x}) = \alpha_0^A(\vec{x}) + \alpha_0^B(\vec{x})$, where A and B are the regions occupied by the two dielectrics. We then have

$$\begin{aligned} F(w) &= \frac{1}{2} \text{Tr} \left(\ln \left(1 - (\alpha_0^A + \alpha_0^B) D'_0 \right) \right) \\ &= \frac{1}{2} \text{Tr} \left(\ln \left(1 - \alpha_0^A D'_0 \right) \right) + \frac{1}{2} \text{Tr} \left(\ln \left(1 - (1 - \alpha_0^A D'_0)^{-1} \alpha_0^B D'_0 \right) \right) \end{aligned} \quad (46)$$

The first term is just the energy $F_A(w) = \frac{1}{2} \text{Tr} \left(\ln \left(1 - \alpha_0^A D'_0 \right) \right)$ of the dielectric in the region A . For the second term we use $(1 - \alpha_0^A D'_0)^{-1} = 1 + (1 - \alpha_0^A D'_0)^{-1} \alpha_0^A D'_0$ and we get

$$F(w) = F_A(w) + F_B(w) + \frac{1}{2} \text{Tr} \left(\ln \left(1 - (1 - \alpha_0^A D'_0)^{-1} \alpha_0^A D'_0 (1 - \alpha_0^B D'_0)^{-1} \alpha_0^B D'_0 \right) \right) \quad (47)$$

where

$$F_B(w) = \frac{1}{2} \text{Tr} \left(\ln \left(1 - \alpha_0^B D'_0 \right) \right). \quad (48)$$

For the last expression in (47) we use

$$(1 - \alpha_0^A(D^0 - G^0))^{-1} \alpha_0^A = (1 - \alpha_A D^0)^{-1} \alpha_A, \quad (49)$$

with $\alpha_A = (1 - \alpha_0^A G^0)^{-1} \alpha_0^A$. Since $\alpha_A D'_0 \alpha_B = \alpha_A D^0 \alpha_B$ for separated regions A and B , we get for the last term in (47)

$$F_{AB}(w) = \frac{1}{2} \text{Tr} \left(\ln \left(1 - \alpha_A D_A \alpha_B D_B \right) \right), \quad (50)$$

where

$$D_A = (1 - D^0 \alpha_A)^{-1} D^0 \quad (51)$$

and similar for D_B . D_A is just the Green's function of the electromagnetic field for only one dielectric in the region A as shall be shown in the next section. Hence we have for the free energy

$$F(w) = F_A(w) + F_B(w) + F_{AB}(w) \quad (52)$$

All the three energies are finite. For the energies F_A and F_B one has to take the finite distance between the atoms into account. F_{AB} is the only term where the distance between the dielectrics enters. Only this energy contributes to the Casimir force. It remains also finite in the limit $a \rightarrow 0$. Then for the polarizations α_A and α_B the Lorentz-Lorenz formula (41) holds.

To calculate the force between two dielectrics we will use the stress tensor of the electromagnetic field. The force is given by a contour integral over a region containing only one of the two bodies. Only the stress outside the dielectrics contribute to the force. It can be calculated as usual from the expression:

$$\begin{aligned} T_{ik}(\vec{x}) &= \frac{1}{2} \delta_{ik} \langle E_l(\vec{x}) E_l(\vec{x})' - \langle E_i(\vec{x}) E_k(\vec{x})' \\ &\quad + \frac{1}{2} \delta_{ik} \langle B_l(\vec{x}) B_l(\vec{x})' - \langle B_i(\vec{x}) B_k(\vec{x})' \end{aligned} \quad (53)$$

The prime again means that the 0-th order in α is suppressed.

IV. BASIC INTEGRAL EQUATION

To calculate the Casimir force from (53) one has to calculate the Green's function of the total electromagnetic field $D_{ij}(\vec{x}, \vec{x}') = \langle E_i(\vec{x}) E_j(\vec{x}') \rangle$ given in (44). Let $\alpha(w, \vec{x}) = \alpha(w) \theta_K(\vec{x})$, where K is the region occupied by the dielectric. The Green's function can be expressed as an integral equation:

$$D_{ij}(\omega, \vec{x}, \vec{x}') = D_{ij}^0(\omega, \vec{x} - \vec{x}') + \alpha(\omega) \int_K d^3y D_{ik}^0(\omega, \vec{x} - \vec{y}) D_{kj}(\omega, \vec{y}, \vec{x}'), \quad (54)$$

where no cut-off is necessary. This is our basic integral equation. It separates the many-body problem from the geometry. The latter only enters in the region of integration K occupied by the dielectric. For simplicity we have assumed one kind of dielectrics and vacuum outside K , the generalization to more dielectrics is straightforward.

From the integral equation one can derive a differential equation for D . From

$$((\omega^2 + \Delta) \delta_{ik} - \partial_i \partial_k) D_{kj}^0(\omega, \vec{x}) = -\omega^2 \delta_{ij}(\vec{x}) \quad (55)$$

one easily derives

$$((\varepsilon(\omega, \vec{x})\omega^2 + \Delta) \delta_{ik} - \partial_i \partial_k) D_{kj}(\omega, \vec{x}, \vec{x}') = -\omega^2 \delta_{ij}(\vec{x} - \vec{x}') \quad (56)$$

The boundary conditions can also be derived from the integral equation (54). In D^0 only the part of the dipole-dipole interaction is responsible for non-continuous boundary conditions. Hence the dipole-dipole approximation is necessary to produce hard boundary conditions and in the same time it is responsible for divergences for connected bodies, if the self-energy contributions are ignored.

A solution of the Casimir problem by solving the differential equation with boundary conditions was given by Lifshitz [7] for simple plain geometry. The solution of the boundary-value problem is much harder than the solution of (54). In principle the linear equation (54) can be solved numerically for any geometry. In the following we are interested in cases where this solution can be obtained analytically. We solve directly the integral equation without the use of any boundary condition.

In the shorthand notation the basic integral equation writes as follows:

$$D = D^0 + \alpha D^0 \theta_K D \quad (57)$$

For a infinite dielectric ($K = \mathbb{R}^3$) the equation can be solved in k -space:

$$D^1 = D^0 + \alpha D^0 D^1 = (1 - \alpha D^0)^{-1} D^0 \quad (58)$$

leading to

$$D_{ij}^1(\omega, \vec{k}) = \frac{1}{\varepsilon} \frac{\varepsilon \omega^2 \delta_{ij} - k_i k_j}{k^2 - \varepsilon \omega^2} = \frac{1}{\varepsilon} D_{ij}^0(\sqrt{\varepsilon} \omega, \vec{k}) \quad (59)$$

Let

$$\theta_{\bar{K}} = 1 - \theta_K \quad (60)$$

Substitute θ_K in the basic integral equation (57) gives

$$D = D^0 + \alpha D^0 D - \alpha D^0 \theta_{\bar{K}} D. \quad (61)$$

Hence

$$(1 - \alpha D_0) D = D^0 - \alpha D^0 \theta_{\bar{K}} D \quad (62)$$

And using (58) we get

$$D = D^1 - \alpha D^1 \theta_{\bar{K}} D \quad (63)$$

the complementary basic equation. To obtain the Casimir force we have to compute

$$\theta_K D \theta_{\bar{K}} = \theta_K D^1 \theta_{\bar{K}} - \alpha \theta_K D^1 \theta_{\bar{K}} D \theta_{\bar{K}}. \quad (64)$$

Here we substitute (57)

$$\theta_K D \theta_{\bar{K}} = \theta_K D^1 \theta_{\bar{K}} - \alpha \theta_K D^1 \theta_{\bar{K}} D^0 \theta_{\bar{K}} - \alpha^2 (\theta_K D^1 \theta_{\bar{K}} D^0) \theta_K D \theta_{\bar{K}}. \quad (65)$$

Now the same quantity as on the l.h.s has appeared, which allows the solution

$$\theta_K D \theta_{\bar{K}} = (1 + \alpha^2 \theta_K D^1 \theta_{\bar{K}} D^0)^{-1} \theta_K D^1 \theta_{\bar{K}} (1 - \alpha D^0 \theta_{\bar{K}}). \quad (66)$$

The above formal solution (66) is very useful in the case where the operator in the inverse operates by simple multiplication:

$$(\theta_K D^1 \theta_{\bar{K}} D^0) \theta_K D^1 \theta_{\bar{K}} = \gamma(w) \theta_K D^1 \theta_{\bar{K}}. \quad (67)$$

As we will see this is indeed true for various simple geometries. Then we obtain from (66) and (57) the solutions

$$\theta_K D \theta_{\bar{K}} = \frac{1}{1 + \alpha^2 \gamma} \theta_K D^1 \theta_{\bar{K}} (1 - \alpha D^0 \theta_{\bar{K}}), \quad (68)$$

$$\theta_{\bar{K}} D \theta_{\bar{K}} = \theta_{\bar{K}} D^0 \theta_{\bar{K}} + \frac{\alpha}{1 + \alpha^2 \gamma} \theta_{\bar{K}} D^0 \theta_K D^1 \theta_{\bar{K}} (1 - \alpha D^0 \theta_{\bar{K}}). \quad (69)$$

V. PLAIN GEOMETRIES

Let us first consider an infinitely thick plate in the region $-\infty < x < 0$. In this case we choose the polarization vectors (11) as follows: Let $\vec{n} = (1, 0, 0)$, then

$$\vec{\xi}_1(\vec{k}) = \frac{\vec{k} \wedge \vec{n}}{|\vec{k} \wedge \vec{n}|} = \frac{1}{p}(0, -k_3, k_2) \quad (70)$$

$$\vec{\xi}_2(\vec{k}) = \frac{\vec{k} \wedge (\vec{k} \wedge \vec{n})}{|\vec{k} \wedge (\vec{k} \wedge \vec{n})|} = \frac{1}{pk}(p^2, -k_2 k_1, -k_3 k_1), \quad (71)$$

where $p^2 = k_2^2 + k_3^2$ and $k^2 = k_1^2 + p^2$. D^0 (27) can be decomposed accordingly

$$D^0 = D_1^0 + D_2^0, \quad (72)$$

$$D_{1ij}^0(x) = \frac{1}{2\pi} \int dk_1 \xi_{1i} \xi_{1j} \frac{w^2}{k_1^2 - s_0^2} e^{ik_1 x}, \quad (73)$$

where $s_0^2 = w^2 - p^2$, $w = i\omega_n$, and

$$D_{2ij}^0(x) = \delta(x)(\xi_{1i} \xi_{1j} - \delta_{ij}) + \frac{1}{2\pi} \int dk_1 \xi_{2i} \xi_{2j} \frac{k^2}{k_1^2 - s_0^2} e^{ik_1 x}. \quad (74)$$

The integrals can be easily computed:

$$D_{1ij}^0(x) = \xi_{1i} \xi_{1j} \frac{iw^2}{2s_0} \left(\theta(x) e^{is_0 x} + \theta(-x) e^{-is_0 x} \right) \quad (75)$$

$$D_{2ij}^0(x) = \delta(x)(\xi_{1i} \xi_{1j} - \delta_{ij}) + \xi_{2i} \xi_{2j} \frac{iw^2}{2s_0} \left(\theta(x) e^{is_0 x} + \theta(-x) e^{-is_0 x} \right), \quad (76)$$

where $\vec{\xi}_2 = 1/wp(p^2, i\partial_x k_2, i\partial_x k_3)$ due to the Fourier transformation. The corresponding formulae for the homogeneous dielectric (59) are similar:

$$D_{1ij}^1(x) = \xi_{1i} \xi_{1j} \frac{iw^2}{2s_1} \left(\theta(x) e^{is_1 x} + \theta(-x) e^{-is_1 x} \right) \quad (77)$$

and

$$D_{2ij}^1(x) = \frac{1}{\varepsilon} \delta(x)(\xi_{1i} \xi_{1j} - \delta_{ij}) + \xi'_{2i} \xi'_{2j} \frac{i\varepsilon w^2}{2s_1} \left(\theta(x) e^{is_1 x} + \theta(-x) e^{-is_1 x} \right), \quad (78)$$

where

$$s_1^2 = \varepsilon w^2 - p^2, \quad \vec{\xi}_2' = \frac{1}{\sqrt{\varepsilon wp}}(p^2, i\partial_x k_2, i\partial_x k_3). \quad (79)$$

The above decomposition into the two transverse modes (TE and TM mode) of the radiation field leads to a similar decomposition in (68) and (69). There is no mode mixing. Details of the calculation are given in Appendix A. For (68) we obtain

$$\theta(-x) D(x, x') \theta(x') = \left(\frac{iw^2}{s_1 + s_0} \xi_{1i} \xi_{1j}^* + \frac{i\sqrt{\varepsilon} w^4}{(s_1 + s_0)(p^2 + s_1 s_0)} \xi_{2i} \xi_{2j}^* \right) \theta(-x) e^{-is_1 x} \theta(x') e^{is_0 x'}, \quad (80)$$

and for the reflection (69)

$$\begin{aligned} \theta(x) D(x, x') \theta(x') &= \theta(x) D^0(x, x') \theta(x') \\ &+ \frac{iw^2}{2s_0} \left(r_1(w, p) \xi_{1i} \xi_{1j}^* + r_2(w, p) \xi_{2i} \xi_{2j}^* \right) \theta(x) e^{is_0 x} \theta(x') e^{is_0 x'} \end{aligned} \quad (81)$$

where the second term herein represents the reflection at the boundary $x = 0$ with the reflection factors r_1 and r_2 for the TE and the TM mode given by

$$r_1(w, p) = -\frac{s_1 - s_0}{s_1 + s_0} = -\frac{\sqrt{\varepsilon(w)w^2 - p^2} - \sqrt{w^2 - p^2}}{\sqrt{\varepsilon(w)w^2 - p^2} + \sqrt{w^2 - p^2}} \quad (82)$$

and

$$r_2(w, p) = -\frac{s_1 - \varepsilon s_0}{s_1 + \varepsilon s_0} = -\frac{\sqrt{\varepsilon(w)w^2 - p^2} - \varepsilon(w)\sqrt{w^2 - p^2}}{\sqrt{\varepsilon(w)w^2 - p^2} + \varepsilon(w)\sqrt{w^2 - p^2}}. \quad (83)$$

For later use we rename the Green's function for the region $A : x < 0$ by $D^A(x, x')$. Now we fill also the region $B : x > a > 0$ with the dielectric with the same ε , in between $0 < x < a$ we have vacuum. The Green's functions for the region B alone are obtained from the above formulae by substituting x by $a - x$:

$$\theta_B D^B \theta_{\bar{B}} = \left(\frac{iw^2}{s_1 + s_0} \xi_{1i} \xi_{1j}^* + \frac{i\sqrt{\varepsilon} w^4}{(s_1 + s_0)(p^2 + s_1 s_0)} \xi_{2i} \xi_{2j}^* \right) \theta(a - x) e^{-is_1(a-x)} \theta(a - x') e^{is_0(a-x')}, \quad (84)$$

$$\begin{aligned} \theta_{\bar{B}} D(x, x') \theta_{\bar{B}} &= \theta(a - x) D^0(x, x') \theta(a - x') \\ &+ \frac{iw^2}{2s_0} \left(r_1(w, p) \xi_{1i} \xi_{1j}^* + r_2(w, p) \xi_{2i} \xi_{2j}^* \right) \theta(a - x) e^{-is_0 x} \theta(a - x') e^{-is_0 x'} e^{2is_0 a}. \end{aligned} \quad (85)$$

According to (57) the total Green's function is the solution of the integral equation

$$D = D^0 + \alpha D^0 \theta_A D + \alpha D^0 \theta_B D. \quad (86)$$

Inserting the definitions of D^A and D^B , we get

$$D = D^A + \alpha D^A \theta_B D \quad (87)$$

$$D = D^B + \alpha D^B \theta_A D. \quad (88)$$

Let us introduce the characteristic function

$$\theta_C = 1 - \theta_A - \theta_B \quad (89)$$

for the region between the two dielectrics. Then by combining the equations we find for the polarization λ :

$$\begin{aligned} \theta_A D_\lambda \theta_C &= \theta_A D_\lambda^A \theta_C + \alpha \theta_A D_\lambda^A \theta_B D_\lambda^B \theta_C + \alpha^2 \theta_A D_\lambda^A \theta_B D_\lambda^B \theta_A D_\lambda \theta_C \\ &= \frac{1}{1 - \alpha^2 \gamma_\lambda} (\theta_A D_\lambda^A \theta_C + \alpha \theta_A D_\lambda^A \theta_B D_\lambda^B \theta_C). \end{aligned} \quad (90)$$

Here we have used the fact that the operator

$$(\theta_A D_\lambda^A \theta_B D_\lambda^B) \theta_A D_\lambda^A \theta_B = \gamma_\lambda \theta_A D_\lambda^A \theta_B \quad (91)$$

operates simply by multiplication, this is shown in Appendix A. Substitution into (88) gives

$$\theta_C D_\lambda \theta_C = \theta_C D_\lambda^B \theta_C + \mu_\lambda \alpha (\theta_C D_\lambda^B \theta_A D_\lambda^A \theta_C + \alpha \theta_C D_\lambda^B \theta_A D_\lambda^A \theta_B D_\lambda^B \theta_C), \quad (92)$$

where

$$\mu_\lambda = \frac{1}{1 - \alpha^2 \gamma_\lambda}. \quad (93)$$

Substitution of D_λ^B by its defining equation $D_\lambda^B = D_\lambda^0 + \alpha D_\lambda^0 \theta_B D_\lambda^B$ leads to

$$\begin{aligned} \theta_C D_\lambda \theta_C &= \theta_C D_\lambda^0 \theta_C + \mu_\lambda \alpha \theta_C D_\lambda^0 \theta_A D_\lambda^A \theta_C + \mu_\lambda \alpha \theta_C D_\lambda^0 \theta_B D_\lambda^B \theta_C \\ &+ \mu_\lambda \alpha^2 \theta_C D_\lambda^0 \theta_A D_\lambda^A \theta_B D_\lambda^B \theta_C + \mu_\lambda \alpha^2 \theta_C D_\lambda^0 \theta_B D_\lambda^B \theta_A D_\lambda^A \theta_C \end{aligned} \quad (94)$$

The calculation of (94) is given in Appendix A with the following results:

$$\begin{aligned} D_\lambda(x, x') &= D_\lambda^0(x, x') \\ &\quad + \mu_\lambda r_\lambda \frac{iw^2}{2s_0} \xi_\lambda \xi_\lambda^* \left(e^{is_0(x+x')} + e^{-is_0(x+x'-2a)} \right) \\ &\quad + \mu_\lambda r_\lambda^2 e^{2is_0 a} \frac{iw^2}{2s_0} \xi_\lambda \xi_\lambda^* \left(e^{is_0(x-x')} + e^{-is_0(x-x')} \right), \end{aligned} \quad (95)$$

$$\mu_\lambda = \frac{1}{1 - r_\lambda^2 e^{2is_0 a}}, \quad (96)$$

with $\lambda = 1, 2$ for the two modes and where r_λ are the reflection factors given in (82) and (83). Since the 0-th order does not contribute later on, we introduce $D^W = D - D^0$:

$$D_{ij}^W(w; x, x') = \xi_{1i}(x) \xi_{1j}(x')^* f^1(w; x, x') + \xi_{2i}(x) \xi_{2j}(x')^* f^2(w; x, x'), \quad (97)$$

where

$$\vec{\xi}_1(\vec{k}) = \frac{1}{p} (0, -k_3, k_2) \quad (98)$$

$$\vec{\xi}_2(\vec{k}) = \frac{1}{wp} (p^2, i\partial_x k_2, i\partial_x k_3). \quad (99)$$

VI. CASIMIR FORCE FOR PLAIN GEOMETRY

Let us assume for a moment that the dielectric A covers the half-plain $-\infty < x < 0$ and dielectric B the region $a < x < R$. Then the force per unit square on the surface of B is equal to the difference of the stress tensors $T_{11}(a-) - T_{11}(R+)$. The term $T_{11}(R+)$ vanishes, but $T_{11}(a-)$ contains contributions from the reflection at $x = R$. These contributions vanish in the limit $R \rightarrow \infty$. Then only the Green's function for arguments between the two dielectrics contributes, which we have calculated in the last section. This argument is necessary because otherwise the stress tensor at infinity gives rise to an infinite "volume force".

The stress tensor for 1-direction is given by

$$\begin{aligned} T_{11}(\vec{x}) &= \frac{1}{2} \langle E_2(\vec{x})^2 + E_3(\vec{x})^2 - E_1(\vec{x})^2 \\ &\quad + B_2(\vec{x})^2 + B_3(\vec{x})^2 - B_1(\vec{x})^2 \rangle'. \end{aligned} \quad (100)$$

To find the Green's functions for the magnetic field, we use the field equations. From

$$\vec{E} = -\partial_t \vec{A} \quad \text{and} \quad \vec{B} = \vec{\nabla} \wedge \vec{A} \quad (101)$$

we get for the Fourier transforms

$$\vec{B}(k) = \frac{1}{\omega} \vec{k} \wedge \vec{E}(k). \quad (102)$$

By (70) (71) we have

$$\frac{1}{\omega} \vec{k} \wedge \vec{\xi}_1(\vec{k}) = \xi_2(\vec{k}) \quad \text{and} \quad \frac{1}{\omega} \vec{k} \wedge \vec{\xi}_2(\vec{k}) = -\xi_1(\vec{k}), \quad (103)$$

using $k^2 = \omega^2$. Now we are ready to calculate the Casimir force. We obtain with (97)

$$T_{11}(\vec{x}) = \frac{1}{2\beta} \sum_m e^{i\omega_m \tau} \frac{1}{(2\pi)^2} \int \frac{d^2 p}{w^2} \left(w^2 + \partial_x \partial_{x'} - p^2 \right) \left(f_1(w; x, x') + f_2(w; x, x') \right) \Big|_{x=x'}, \quad (104)$$

where the f_j are defined in (97). Only the terms depending on the difference $(x - x')$ in f_j contribute because they are proportional to

$$w^2 + s_0^2 - p^2 = 2s_0^2. \quad (105)$$

This leads to the following final result

$$T_{11} = \frac{1}{\beta} \sum_{\omega_m} \frac{1}{(2\pi)^2} \int d^2 p i s_0 \left(\frac{r_1^2 e^{2is_0 a}}{1 - r_1^2 e^{2is_0 a}} + \frac{r_2^2 e^{2is_0 a}}{1 - r_2^2 e^{2is_0 a}} \right), \quad (106)$$

where r_1 and r_2 are the reflection factors for the two modes given by (82) and (83). The result (106) is the Lifshitz formula [4] [7].

VII. SPHERICAL GEOMETRY

As mentioned in Sect.3 for compact dielectrics the energy E_D is always divergent, since it ignores the self-energy contribution E_G in equation (37). Nevertheless the above method to calculate the photon Green's function still works for spherical geometries as shall be shown in this section. As a consequence, it is not hard to compute the Casimir force on the spherical shell and rederive the result of ref. [10].

The region occupied by the dielectric is now the sphere

$$K = \{\vec{x} \in \mathbb{R}^3 \mid \vec{x}^2 \leq R^2\}. \quad (107)$$

The free Green's function (23) (24) is then written in spherical coordinates (r, ϑ, φ) using the expansion

$$e^{i\vec{k} \cdot \vec{x}} = 4\pi \sum_{lm} i^l j_l(kr) Y_l^m(\frac{\vec{x}}{r}) Y_l^m(\frac{\vec{k}}{k})^*, \quad (108)$$

where j_l are spherical Bessel functions and Y_l^m the spherical harmonics. For $\vec{x} \neq \vec{x}'$ we then have

$$D_{jj'}^0(w, \vec{x}, \vec{x}') = \frac{2}{\pi} \sum_{\lambda=1,2} \xi_{\lambda j}(\vec{x}) \xi_{\lambda j'}(\vec{x}') \int_0^\infty dk \frac{k^4}{k^2 - w^2} \sum_{lm} j_l(kr) j_l(kr') Y_l^m(\vartheta, \varphi) Y_l^m(\vartheta', \varphi')^*. \quad (109)$$

The k -integral can be extended to $(-\infty, +\infty)$ and then computed using the theorem of residues, taking $w = i\omega_n$ (22) into account. The result is

$$\frac{2}{\pi} \int_0^\infty \dots = iw^3 [\theta(r - r') h_l^{(1)}(wr) j_l(wr') + \theta(r' - r) j_l(wr) h_l^{(1)}(wr')], \quad (110)$$

where $h_l^{(1)}$ are the Bessel functions of the third kind. The two transversal polarization vectors perpendicular to $\vec{k} \sim \vec{p} = -i\partial/\partial\vec{x}$ are constructed as follows: First we choose

$$\vec{\xi}_1(\vec{x}) = \frac{1}{\sqrt{l(l+1)}} \vec{x} \wedge \vec{p} = \frac{1}{\sqrt{l(l+1)}} \vec{L}. \quad (111)$$

This is the angular momentum operator which gives the so-called electric TE mode; it is normalized according to

$$\vec{\xi}_1^2 = \frac{1}{l(l+1)} \vec{L}^2. \quad (112)$$

The second polarization vector is then

$$\vec{\xi}_2(\vec{x}) = \frac{1}{w\sqrt{l(l+1)}} \vec{p} \wedge \vec{L}. \quad (113)$$

This is the TM mode which is also normalized because $|\vec{k}| = w$. Since the metric tensor in spherical coordinates is non-trivial, it is convenient to distinguish between upper and lower tensor components. Let $Q = l(l+1)$. In components we have

$$\begin{aligned}\xi_1^i(r, \phi, \theta) &= \frac{1}{\sqrt{Q}} \frac{1}{r \sin \theta} \left(0, -\partial_\theta, \partial_\phi \right) \\ \xi_{1i}(r, \phi, \theta) &= \frac{1}{\sqrt{Q}} r \left(0, -\sin \theta \partial_\theta, \frac{1}{\sin \theta} \partial_\phi \right)\end{aligned}\quad (114)$$

$$\begin{aligned}\xi_2^i(r, \phi, \theta) &= \frac{1}{k\sqrt{Q}} \left(\frac{Q}{r}, \frac{\partial_r r}{r^2 \sin^2 \theta} \partial_\phi, \frac{\partial_r r}{r^2} \partial_\theta \right) \\ \xi_{2i}(r, \phi, \theta) &= \frac{1}{k\sqrt{Q}} \left(\frac{Q}{r}, \partial_r r \partial_\phi, \partial_r r \partial_\theta \right)\end{aligned}\quad (115)$$

We have orthogonality in the sense [10]

$$\int \left(\xi_1^i(\vec{x}) f(r) Y_l^m(\frac{\vec{x}}{r}) \right)^* \xi_{1i}(\vec{x}) g(r) Y_{l'}^{m'}(\frac{\vec{x}}{r}) d^3x = \delta_{ll'} \delta_{mm'} \int_0^\infty dr r^2 f^*(r) g(r) \quad (116)$$

$$\int \left(\xi_2^i(\vec{x}) f(r) Y_l^m(\frac{\vec{x}}{r}) \right)^* \xi_{2i}(\vec{x}) g(r) Y_{l'}^{m'}(\frac{\vec{x}}{r}) d^3x = \delta_{ll'} \delta_{mm'} \frac{1}{w^2} \int_0^\infty dr \left[\frac{d}{dr} (rf^*) \frac{d}{dr} (rg) + l(l+1)f^*g \right], \quad (117)$$

where we sum over the lower and upper indices i . The mixed expressions are 0. This has the important consequence that different l 's and polarizations don't mix. Therefore, only the radial integrations are non-trivial. The free Green's function (109) is now equal to

$$D_j^{0j'}(w, \vec{x}, \vec{x}') = \sum_{\lambda=1,2} \xi_{\lambda j}(\vec{x}) \xi_\lambda^{j'}(\vec{x}') \sum_{lm} Y_l^m(\vartheta, \varphi) Y_l^m(\vartheta', \varphi')^* D^0(r, r'), \quad (118)$$

where

$$D^0(r, r') = iw^3 \left[\theta(r - r') h_l^{(1)}(wr) j_l(wr') + \theta(r' - r) h_l^{(1)}(wr') j_l(wr) \right], \quad (119)$$

and for the infinite dielectric we have

$$D^1(w, \vec{x}, \vec{x}') = \frac{1}{\varepsilon} D^0(\sqrt{\varepsilon}w, \vec{x}, \vec{x}'). \quad (120)$$

The essential point in the solution of the problem is the calculation of (67). For the TE mode and $r, r' < R$ we get

$$\begin{aligned}\theta_K D^1 \theta_{\bar{K}} D^0 \theta_K |_l(r, r') &= \int_R^\infty d\varrho \varrho^2 D^1(r, \varrho) D^0(\varrho, r') \\ &= -\langle h_l^{(1)}, h_l^{(1)} \rangle_1 \sqrt{\varepsilon} w^6 j_l(\sqrt{\varepsilon} wr) j_l(wr'),\end{aligned}\quad (121)$$

where we have introduced the non-symmetric bilinear form of Bessel functions:

$$\begin{aligned}\langle f_l, g_l \rangle_1 &= \int_0^R d\varrho \varrho^2 f_l(w\varrho) g_l(\sqrt{\varepsilon} w\varrho) \\ &= \frac{R}{\alpha w^2} \left[wf'_l(wR) g_l(\sqrt{\varepsilon} wR) - \sqrt{\varepsilon} w f_l(wR) g'_l(\sqrt{\varepsilon} wR) \right].\end{aligned}\quad (122)$$

Then (67) operates multiplicatively, indeed,

$$(\theta_K D^1 \theta_{\bar{K}} D^0) \theta_K D^1 \theta_{\bar{K}} |_l = \gamma_{1l} \theta_K D^1 \theta_{\bar{K}} |_l, \quad (123)$$

where

$$\gamma_{1l} = \sqrt{\varepsilon} w^6 \langle h_l^{(1)}, h_l^{(1)} \rangle_1 \langle j_l, j_l \rangle_1. \quad (124)$$

Since

$$(\theta_K D^1 \theta_{\bar{K}} - \alpha \theta_K D^1 \theta_{\bar{K}} D^0 \theta_{\bar{K}}) \big|_l = -\alpha \sqrt{\varepsilon} w^6 \langle j_l, h_l^{(1)} \rangle_1 j_l(\sqrt{\varepsilon} wr) h_l^{(1)}(wr'), \quad (125)$$

(see also (A11) in the appendix) we obtain for (68)

$$\begin{aligned} \theta_K D_j^{j'} \theta_{\bar{K}} \big|_{\lambda=1} &= -\frac{1}{1 + \alpha^2 \gamma_1} \alpha \sqrt{\varepsilon} w^6 \langle j_l, h_l^{(1)} \rangle_1 \\ &\times \xi_{1,j}(\vec{x}) \xi_1^{j'}(\vec{x}') Y_l^m(\vartheta, \varphi) Y_l^m(\vartheta', \varphi') j_l(\sqrt{\varepsilon} wr) h_l^{(1)}(wr') \end{aligned} \quad (126)$$

for $\lambda = 1$ and similarly for $\lambda = 2$

$$\begin{aligned} \theta_K D_j^{j'} \theta_{\bar{K}} \big|_{\lambda=2} &= -\frac{1}{1 + \alpha^2 \gamma_2} \alpha \frac{w^2}{\sqrt{\varepsilon}} \langle j_l, h_l^{(1)} \rangle_2 \\ &\times \xi_{2,j}(\vec{x}) \xi_2^{j'}(\vec{x}') Y_l^m(\vartheta, \varphi) Y_l^m(\vartheta', \varphi') j_l(\sqrt{\varepsilon} wr) h_l^{(1)}(wr') , \end{aligned} \quad (127)$$

with

$$\gamma_2 = \frac{w^2}{\sqrt{\varepsilon}} \langle j_l, j_l \rangle_2 \langle h_l^{(1)}, h_l^{(1)} \rangle_2, \quad (128)$$

where we have introduced a second bilinear form for Bessel functions

$$\begin{aligned} \langle f_l, g_l \rangle_2 &= \int_0^R dr \left[\frac{d}{dr} \left(r f_l(wr) \right) \frac{d}{dr} \left(r g_l(\sqrt{\varepsilon} wr) \right) + l(l+1) f_l(wr) g_l(\sqrt{\varepsilon} wr) \right] \\ &= \left[\frac{d}{dR} \left(R f_l(wR) \right) \right] R g_l(\sqrt{\varepsilon} wR) + w^2 \langle f_l, g_l \rangle_1. \end{aligned} \quad (129)$$

For (69) we obtain:

$$\theta_{\bar{K}} D_j^{j'} \theta_{\bar{K}} \big|_{\lambda} = D_j^{0,j'} \big|_{\lambda} + \sum_{lm} \mu_{\lambda l} i w^3 \xi_{\lambda j}(\vec{x}) \xi_{\lambda}^{j'}(\vec{x}') Y_l^m(\vartheta, \varphi) Y_l^m(\vartheta', \varphi') h_l^{(1)}(wr) h_l^{(1)}(wr'), \quad (130)$$

where for $\lambda = 1$

$$\mu_{1l} = -\frac{\alpha^2 \sqrt{\varepsilon} w^6 \langle j_l, j_l \rangle_1 \langle j_l, h_l^{(1)} \rangle_1}{1 + \alpha^2 \sqrt{\varepsilon} w^6 \langle j_l, j_l \rangle_1 \langle h_l^{(1)}, h_l^{(1)} \rangle_1}, \quad (131)$$

and similarly for $\lambda = 2$:

$$\mu_{2l} = -\frac{\alpha^2 \frac{w^2}{\sqrt{\varepsilon}} \langle j_l, j_l \rangle_2 \langle j_l, h_l^{(1)} \rangle_2}{1 + \alpha^2 \frac{w^2}{\sqrt{\varepsilon}} \langle j_l, j_l \rangle_2 \langle h_l^{(1)}, h_l^{(1)} \rangle_2}. \quad (132)$$

VIII. COMMENTS ON THE METALLIC LIMIT

In this section the metallic limit $\varepsilon \rightarrow 0$ is considered. By the Lorentz-Lorenz relation (41) this means to put the atomic polarization $\alpha_0 = 3$. It is not clear whether in our microscopic theory the perturbation expansion for the Green's function converges for $\alpha_0 = 3$. Our microscopic theory does not give an adequate description for metals. In fact we'll show below that in infinite homogeneous dielectrics the series only converges for $\alpha_0 < \frac{3}{2}$. Nevertheless for the Casimir effect only Green's function that crosses the boundaries contributes to the force. Indeed the series then converges in the TE mode, at least, for small frequencies $p^2 > 2u^2$, where $u = w_n = -iw$. We'll show this only for the special case of flat geometries.

The Green's function of the radiation field is given by

$$D = D^0 + \alpha_0 D^0 \theta_A D^0 + \alpha_0^2 D^0 \theta_A D' \theta_A D^0, \quad (133)$$

where

$$D' = D'_0 + \alpha_0 D'_0 \theta_A D'_0 + \alpha_0^2 (D'_0 \theta_A)^2 D'_0 + \dots \quad (134)$$

with $D'_0 = D^0 + \frac{1}{3}$. In momentum-space D'_0 is given by

$$\begin{aligned} D'_0(iu, \vec{k}) &= \frac{-u^2 \delta_{ij} - k_i k_j}{k^2 + u^2} + \frac{1}{3} \delta_{ij} \\ &= \frac{1}{3} \frac{-2u^2 + k^2}{k^2 + u^2} \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) - \frac{2}{3} \frac{k_i k_j}{k^2} \end{aligned} \quad (135)$$

It is clear that in infinite homogeneous dielectrics the perturbation expansion only converges for $\alpha_0 < \frac{3}{2}$ because of the dipole-dipole interaction.

For the Green's function that crosses the boundaries, let us first consider the case $u = 0$. Then only the dipole-dipole interaction contributes. We have

$$D'_0(0, \vec{k}) = -\frac{k_i k_j}{k^2} + \frac{1}{3} \delta_{ij} = D_0^l(\vec{k}) + \frac{1}{3} \delta_{ij} \quad (136)$$

Let A be the region of the right half-plane with $0 < x < +\infty$. The longitudinal Green's function is given by

$$D_0^l(x - y) = -(-i\partial_x, p_2, p_3)^T (i\partial_y, p_2, p_3) d^0(x - y), \quad (137)$$

where

$$d^0(x) = \frac{1}{2\pi} \int dk_1 \frac{1}{k_1^2 + p^2} e^{ik_1 x} = \frac{1}{2p} (\theta(x)e^{-px} + \theta(-x)e^{px}), \quad (138)$$

with $p = \sqrt{p_2^2 + p_3^2}$. Then a straight-forward calculation gives

$$\theta_A D_0^l \theta_A D_0^l \theta_{\bar{A}} = -\frac{1}{2} \theta_A D_0^l \theta_{\bar{A}} \quad (139)$$

For the Green's function $\theta_A D \theta_{\bar{A}}$ we get

$$\theta_A D \theta_{\bar{A}} = \left(1 - \frac{\alpha_0}{2} \left(1 - \frac{\alpha_0}{6} + \left(\frac{\alpha_0}{6} \right)^2 + \dots \right) \right) \theta_A D_0^l \theta_{\bar{A}} \quad (140)$$

This is convergent for $\alpha_0 < 6$. Hence we get

$$\theta_A D \theta_{\bar{A}} = \left(1 - \frac{\alpha_0}{2} \frac{1}{1 + \frac{\alpha_0}{6}} \right) \theta_A D_0^l \theta_{\bar{A}} \quad (141)$$

This is 0 for $\alpha_0 = 3$ as it should for metals. For the reflection we get with $(\theta_{\bar{A}} D_0^l \theta_A D_0^l \theta_{\bar{A}})(x, x') = -\frac{1}{2} D_0^l(x + x')$ for $x, x' > 0$

$$D(x, x') = D_0^l(x - x') - \frac{\alpha_0}{2} \frac{1}{1 + \frac{\alpha_0}{6}} D_0^l(x + x') = D_0^l(x - x') - \frac{\varepsilon - 1}{\varepsilon + 1} D_0^l(x + x') \quad (142)$$

where

$$\varepsilon = 1 + \frac{\alpha_0}{1 - \frac{\alpha_0}{3}}. \quad (143)$$

For $u > 0$ it is much harder to show that series converges in the metallic limit. We will show this only for the special case where $p^2 > 2u^2 > 0$ and only for TE mode. Let the atomic polarization be any function with

$$\alpha_0(iu) < 3 \quad \text{for } u > 0, \quad \alpha_0(0) = 3. \quad (144)$$

We express the expansion for the Green's function D' (134) formally by the integral equation

$$D' = D'_0 + \alpha_0 D'_0 \theta_A D', \quad (145)$$

where the TE mode of the free Green's function D'_0 is given by

$$D'_0(iu, \vec{k}) = D^0(u, \vec{k}) + \frac{1}{3} = \frac{-u^2}{k^2 + u^2} + \frac{1}{3}. \quad (146)$$

For $p^2 = k_2^2 + k_3^2 > 2u^2 > 0$ the expansion for Green's function D'_1 in a infinite homogeneous dielectric converges and is given by

$$D'_1(iu, \vec{k}) = \frac{\alpha}{\alpha_0} + \frac{\alpha^2}{\alpha_0^2} D^1(u, \vec{k}), \quad (147)$$

with

$$D^1(iu, \vec{k}) = \frac{-u^2}{k^2 + \varepsilon u^2}. \quad (148)$$

This leads to the complementary integral equation

$$D' = D'_1 - \alpha_0 D'_1 \theta_{\bar{A}} D'. \quad (149)$$

We then get

$$\begin{aligned} \theta_A D' \theta_{\bar{A}} &= \theta_A D^0 \theta_{\bar{A}} + \alpha_0 \theta_A D^0 \theta_A D' \theta_{\bar{A}} + \frac{\alpha_0}{3} \theta_A D' \theta_{\bar{A}} \\ &= \theta_A D^0 \theta_{\bar{A}} + \alpha_0 \theta_A D^0 D' \theta_{\bar{A}} - \alpha_0 \theta_A D^0 \theta_{\bar{A}} D' \theta_{\bar{A}} + \frac{\alpha_0}{3} \theta_A D' \theta_{\bar{A}}. \end{aligned} \quad (150)$$

Substitution of D' (149) into the second term of the second line and use of $D^0 + \alpha_0 D^0 D'_1 = \frac{\alpha}{\alpha_0} D^1$ gives

$$\begin{aligned} \theta_A D' \theta_{\bar{A}} &= \frac{\alpha}{\alpha_0} \theta_A D^1 \theta_{\bar{A}} - \alpha \theta_A D^1 \theta_{\bar{A}} D' \theta_{\bar{A}} + \frac{\alpha_0}{3} \theta_A D' \theta_{\bar{A}} \\ &= \frac{\alpha}{\alpha_0} \theta_A D^1 \theta_{\bar{A}} - \alpha \theta_A D^1 \theta_{\bar{A}} D^0 \theta_{\bar{A}} - \alpha_0 \alpha \theta_A D^1 \theta_{\bar{A}} D^0 \theta_A D' \theta_{\bar{A}} + \frac{\alpha_0}{3} \theta_A D' \theta_{\bar{A}} \end{aligned} \quad (151)$$

Now the operator

$$(-\alpha_0 \alpha \theta_A D^1 \theta_{\bar{A}} D^0 \theta_A + \frac{\alpha_0}{3}) \theta_A D^1 \theta_{\bar{A}} = \gamma \theta_A D^1 \theta_{\bar{A}} \quad (152)$$

operates by a simple multiplication. From (A9) in the appendix we get

$$\gamma = \frac{\alpha_0}{3} - \frac{\alpha_0(s_1 - s_0)^2}{\alpha_0 s_1 s_0} < 1 \quad (153)$$

for $u > 0$, hence the perturbation expansion converges. The reflection factor for the TE mode is then given by (82)

$$r_1(u, p) = -\frac{s_1 - s_0}{s_1 + s_0} = -\frac{\sqrt{\varepsilon u^2 + p^2} - \sqrt{u^2 + p^2}}{\sqrt{\varepsilon u^2 + p^2} + \sqrt{u^2 + p^2}}. \quad (154)$$

This formula for the reflection for the TE mode only holds for $p^2 > 2u^2 > 0$. For $u = 0$ the TE mode does not contribute to the reflection and to the free energy, independent of the model adopted for the dielectric function.

This cannot be deduced directly from the Lifshitz formula, where the TE mode does not contribute only if $\lim_{u \rightarrow 0} \varepsilon(iu)u^2 = 0$, which does not hold in the plasma model for metals where

$$\alpha_0(iu) = \frac{u_p^2}{u^2 + \frac{u_p^2}{3}} \longrightarrow \varepsilon(iu) = 1 + \frac{u_p^2}{u^2}. \quad (155)$$

Although $\lim_{u \rightarrow 0} r_1(u, p) \neq 0$ for the plasma model, in our microscopic theory the TE mode does not contribute to the free energy for $u = 0$.

A different result is also obtained for the TM mode for the reflection at a finite metallic plate in the region $0 > x > -R$. It can be seen from formula (155), that for $u \rightarrow 0$ the atomic polarization $\alpha_0(0) = 3$ does not depend from the plasma frequency u_p . This has the consequence that as can be shown, that the reflection at the plate does not depend on the thickness R of the plate for $u = 0$ and we get formula (142) with $\alpha_0 = 3$. Whereas in the macroscopic theory there is a contribution from the reflection at $x = R$.

The small frequency behavior is important for the temperature dependence of the Casimir force, that is widely discussed in recent papers [11–17].

APPENDIX A:

In this Appendix we do the main calculations for the plain geometries. All the calculations are simple since it suffices to calculate surface integrals. First we have to compute the reflection at one plate. We use formula (66)

$$\theta_K D \theta_K = (1 + \alpha^2 \theta_K D^1 \theta_{\bar{K}} D^0)^{-1} \theta_K D^1 \theta_{\bar{K}} (1 - \alpha D^0 \theta_{\bar{K}}). \quad (\text{A1})$$

To compute the integral

$$\int_{\bar{K}} d^3y \theta_K(\vec{x}) D^1(\vec{x} - \vec{y}) D^0(\vec{y} - \vec{z}) \theta_K(\vec{z}) \quad (\text{A2})$$

we use the differential equations

$$\begin{aligned} (\Delta + \omega^2) D_{ij}^0(\vec{y}) &= -(\partial_i \partial_j + \omega^2 \delta_{ij}) \delta(\vec{y}) \\ (\Delta + \varepsilon \omega^2) D_{ij}^1(\vec{y}) &= -\frac{1}{\varepsilon} (\partial_i \partial_j + \varepsilon \omega^2 \delta_{ij}) \delta(\vec{y}). \end{aligned} \quad (\text{A3})$$

On the r.h.s is a function with point support. Since in (A2) \vec{y} and \vec{x} resp. \vec{z} have different support D^0 and D^1 simply solve the wave equations. By Green's theorem we get a surface integral

$$\begin{aligned} \int_{\bar{K}} d^3y \theta_K(\vec{x}) D^1(\vec{x} - \vec{y}) D^0(\vec{y} - \vec{z}) \theta_K(\vec{z}) &= \frac{1}{\alpha \omega^2} \int_{\partial K} d^2 \vec{\sigma}(y) \theta_K(\vec{x}) \\ &\times \left((\vec{\nabla}_{\vec{y}} D^1(\vec{x} - \vec{y})) D^0(\vec{y} - \vec{z}) - D^1(\vec{x} - \vec{y}) \vec{\nabla}_{\vec{y}} D^0(\vec{y} - \vec{z}) \right) \theta_K(\vec{z}) \end{aligned} \quad (\text{A4})$$

where the surface vector $\vec{\sigma}(\vec{y})$ shows inside the region K . Because of the hight symmetry of the plain or the spherical geometries no integral at all has to be computed.

For plain geometries and A representing the region $x < 0$ we get

$$\theta_A D_1^1 \theta_{\bar{A}} D_1^0 \theta_A = \frac{i \omega^2 (s_1 - s_0)}{\alpha 4 s_1 s_0} \xi_{1i} \xi_{1j}^* \theta(-x) e^{-is_1 x} \theta(-x') e^{-is_0 x'} \quad (\text{A5})$$

for the electric TE mode and

$$\theta_A D_2^1 \theta_{\bar{A}} D_2^0 \theta_A = \frac{i \omega^2 (s_1 - s_0) (p^2 - s_1 s_0)}{\alpha 4 s_1 s_0 \sqrt{\varepsilon \omega^2}} \xi_{2i} \xi_{2j}^* \theta(-x) e^{-is_1 x} \theta(-x') e^{-is_0 x'} \quad (\text{A6})$$

for the magnetic TM mode. Similarly we get

$$\theta_A D_1^1 \theta_{\bar{A}} D_1^0 \theta_A D_1^1 \theta_{\bar{A}} = \frac{(s_1 - s_0)^2}{\alpha^2 4 s_1 s_0} \times \frac{i \omega^2}{2 s_1} \xi_{1i} \xi_{1j}^* \theta(-x) \theta(x') e^{-is_1 (x-x')} \quad (\text{A7})$$

for the TE mode and

$$\theta_A D_2^1 \theta_{\bar{A}} D_2^0 \theta_A D_2^1 \theta_{\bar{A}} = \frac{(s_1 - s_0)^2 (p^2 - s_1 s_0)^2}{\alpha^2 \varepsilon \omega^4 4 s_1 s_0} \times \frac{i \omega^2}{2 s_1} \xi_{2i} \xi_{2j}^* \theta(-x) \theta(x') e^{-is_1 (x-x')} \quad (\text{A8})$$

for the TM mode. Hence $\theta_A D^1 \theta_{\bar{A}} D^0 \theta_A$ operates by a simple multiplication on $\theta_A D^1 \theta_{\bar{A}}$ with the factors

$$\gamma_1 = \frac{(s_1 - s_0)^2}{\alpha^2 4 s_1 s_0} \quad \text{and} \quad \gamma_2 = \frac{(s_1 - s_0)^2 (p^2 - s_1 s_0)^2}{\alpha^2 \varepsilon \omega^4 4 s_1 s_0} \quad (\text{A9})$$

for the two modes. Now we compute formula (68)

$$\theta_A D \theta_{\bar{A}} = \frac{1}{1 + \alpha^2 \gamma} (\theta_A D^1 \theta_{\bar{A}} - \alpha \theta_A D^1 \theta_{\bar{A}} D^0 \theta_A). \quad (\text{A10})$$

The expression in the bracket can again be computed by the use of the differential equations (A3). The Dirac functions in the differential equations eliminates the first part. With Green's theorem we obtain

$$\begin{aligned} &\theta_A D^1 \theta_{\bar{A}} - \alpha \theta_A D^1 \theta_{\bar{A}} D^0 \theta_A \\ &= \frac{\alpha}{\alpha \omega^2} \int_{\bar{A}} d^3y \theta_A(\vec{x}) (\Delta_y D^1(\vec{x} - \vec{y}) D^0(\vec{y} - \vec{x}') - D^1(\vec{x} - \vec{y}) \Delta_y D^0(\vec{y} - \vec{x}')) \theta_{\bar{A}}(\vec{x}') \\ &= -\frac{1}{\omega^2} \int_{\partial A} d^2 \vec{\sigma}(\vec{y}) \theta_A(\vec{x}) \left(\vec{\nabla}_y D^1(\vec{x} - \vec{y}) D^0(\vec{y} - \vec{x}') - D^1(\vec{x} - \vec{y}) \vec{\nabla}_y D^0(\vec{y} - \vec{x}') \right) \theta_{\bar{A}}(\vec{x}') \end{aligned} \quad (\text{A11})$$

For plain geometry we obtain

$$\theta_A D_1^1 \theta_{\bar{A}} (1 - D_1^0 \theta_{\bar{A}}) = \frac{i\omega^2(s_1 + s_0)}{4s_1 s_0} \xi_{1i} \xi_{1j}^* \theta(-x) e^{-is_1 x} \theta(x') e^{is_0 x'} \quad (\text{A12})$$

and

$$\theta_A D_2^1 \theta_{\bar{A}} (1 - D_2^0 \theta_{\bar{A}}) = \frac{i\omega^2(s_1 + s_0)(p^2 + s_1 s_2)}{4s_1 s_0 \sqrt{\varepsilon} \omega^2} \xi_{2i} \xi_{2j}^* \theta(-x) e^{-is_1 x} \theta(x') e^{is_0 x'}. \quad (\text{A13})$$

The factors in (A10) simplifies to

$$\frac{1}{1 + \alpha^2 \gamma_1} = \frac{4s_1 s_0}{(s_1 + s_0)^2} \quad \text{and} \quad \frac{1}{1 + \alpha^2 \gamma_2} = \frac{4s_1 s_0 \varepsilon \omega^4}{(s_1 + s_0)^2 (p^2 + s_1 s_0)^2}, \quad (\text{A14})$$

where in the second factor $\varepsilon \omega^4 = (s_1^2 + p^2)(s_0^2 + p^2)$ was used in the calculation. Hence we get for (A1)

$$\theta(-x) D^A(x, x') \theta(x') = \left(\frac{iw^2}{s_1 + s_0} \xi_{1i} \xi_{1j}^* + \frac{i\sqrt{\varepsilon} w^4}{(s_1 + s_0)(p^2 + s_1 s_0)} \xi_{2i} \xi_{2j}^* \right) \theta(-x) e^{-is_1 x} \theta(x') e^{is_0 x'} \quad (\text{A15})$$

For the reflection at one plate we have to compute (69):

$$\theta_{\bar{A}} D \theta_{\bar{A}} = \theta_{\bar{A}} D^0 \theta_{\bar{A}} + \frac{\alpha}{1 + \alpha^2 \gamma} \theta_{\bar{A}} D^0 \theta_A D^1 \theta_{\bar{A}} (1 - \alpha D^0 \theta_{\bar{A}}). \quad (\text{A16})$$

The further integration adds the factors $-\frac{s_1 - s_0}{2s_0}$ and $-\frac{(s_1 - s_0)(p^2 - s_1 s_0)}{2s_0 \sqrt{\varepsilon} \omega^2}$. We obtain the result (81) for the reflection at one plate:

$$\theta(x) D^A(x, x') \theta(x') = \theta(x) D^0(x, x') \theta(x') + \frac{iw^2}{2s_0} \left(r_1(w, p) \xi_{1i} \xi_{1j}^* + r_2(w, p) \xi_{2i} \xi_{2j}^* \right) \theta(x) e^{is_0 x} \theta(x') e^{is_0 x'} \quad (\text{A17})$$

with the two reflection factors

$$r_1 = -\frac{s_1 - s_0}{s_1 + s_0} \quad \text{and} \quad r_2 = -\frac{(s_1 - s_0)(p^2 - s_1 s_0)}{(s_1 + s_0)(p^2 + s_1 s_0)}. \quad (\text{A18})$$

The reflection factor r_2 for the TM mode simplifies with $s_1^2 = \varepsilon w^2 - p^2$ and $s_0^2 = w^2 - p^2$ to

$$r_2 = -\frac{s_1 - \varepsilon s_0}{s_1 + \varepsilon s_0} \quad (\text{A19})$$

For a plate in the region B with $x > a$ we just have to replace in the formulae x by $a - x$. We will just need the formula for $\theta_B D^B \theta_{\bar{B}}$. From (A15) we get

$$\theta(x - a) D^B(x, x') \theta(a - x') = \left(\frac{iw^2}{s_1 + s_0} \xi_{1i} \xi_{1j}^* + \frac{i\sqrt{\varepsilon} w^4}{(s_1 + s_0)(p^2 + s_1 s_0)} \xi_{2i} \xi_{2j}^* \right) \theta(x - a) e^{is_1(x-a)} \theta(a - x') e^{is_0(a-x')}. \quad (\text{A20})$$

For the Green's function for two parallel plates we use formula (94)

$$\begin{aligned} \theta_C D_\lambda \theta_C &= \theta_C D_\lambda^0 \theta_C + \mu_\lambda \alpha \theta_C D_\lambda^0 \theta_A D_\lambda^A \theta_C + \mu_\lambda \alpha \theta_C D_\lambda^0 \theta_B D_\lambda^B \theta_C \\ &\quad + \mu_\lambda \alpha^2 \theta_C D_\lambda^0 \theta_A D_\lambda^A \theta_B D_\lambda^B \theta_C + \mu_\lambda \alpha^2 \theta_C D_\lambda^0 \theta_B D_\lambda^B \theta_A D_\lambda^A \theta_C \end{aligned} \quad (\text{A21})$$

where

$$\mu_\lambda = \frac{1}{1 - \alpha^2 \gamma_\lambda}. \quad (\text{A22})$$

and γ_λ is given by

$$(\theta_A D_\lambda^A \theta_B D_\lambda^B) \theta_A D_\lambda^A \theta_B = \gamma_\lambda \theta_A D_\lambda^A \theta_B \quad (\text{A23})$$

First we compute the expression in the quote in (A23). For D^A and D^B similar differential equation holds as in (A3), so that we just have to compute a surface integral giving

$$\theta_A D_1^A \theta_B D_1^B \theta_A = \frac{-i(s_1 - s_0)w^2}{\alpha(s_1 + s_0)^2} e^{i2s_0a} \xi_1 \xi_1^* \theta(-x) e^{-is_1x} \theta(-x') e^{-is_0x'} \quad (\text{A24})$$

for the TE mode and

$$\theta_A D_2^A \theta_B D_2^B \theta_A = \frac{-i\sqrt{\varepsilon}\omega^4(s_1 - s_0)(p^2 - s_1s_0)}{\alpha(s_1 + s_0)^2(p^2 + s_1s_0)^2} e^{i2s_0a} \xi_2 \xi_2^* \theta(-x) e^{-is_1x} \theta(-x') e^{-is_0x'} \quad (\text{A25})$$

for the TM mode. A further integration adds the factors $\frac{i(s_1 - s_0)}{\alpha\omega^2}$ and $\frac{i(s_1 - s_0)(p^2 - s_1s_0)}{\alpha\sqrt{\varepsilon}\omega^4}$, so that we get for γ_λ in (A23)

$$\alpha^2 \gamma_1 = \frac{(s_1 - s_0)^2}{(s_1 + s_0)^2} e^{i2s_0a} \quad \text{and} \quad \alpha^2 \gamma_2 = \frac{(s_1 - s_0)^2(p^2 - s_1s_0)^2}{(s_1 + s_0)^2(p^2 + s_1s_0)^2} e^{i2s_0a} \quad (\text{A26})$$

where the reflection factors from (A18) appear again. For μ_λ we get

$$\mu_\lambda = \frac{1}{1 - r_\lambda^2 e^{2is_0a}} \quad (\text{A27})$$

Finally we have to compute (A21). The second term is just the reflection at the plate A and is given (A17). The third term is the reflection at the plate B , where x has to be substituted by $a - x$. For the forth and fifth terms the further integration adds again the factors $\frac{i(s_1 - s_0)}{\alpha\omega^2}$ and $\frac{i(s_1 - s_0)(p^2 - s_1s_0)}{\alpha\sqrt{\varepsilon}\omega^4}$ to (A24) and (A25). This gives for x and x' between the two plates

$$\begin{aligned} D_\lambda(x, x') &= D_\lambda^0(x - x') \\ &\quad + \mu_\lambda r_\lambda \frac{i\omega^2}{2s_0} \xi_\lambda \xi_\lambda^* \left(e^{is_0(x+x')} + e^{is_0(2a-x-x')} \right) \\ &\quad + \mu_\lambda r_\lambda^2 e^{2is_0a} \frac{i\omega^2}{2s_0} \xi_\lambda \xi_\lambda^* \left(e^{is_0(x-x')} + e^{-is_0(x-x')} \right) \end{aligned} \quad (\text{A28})$$

for $\lambda = 1, 2$ and with μ_λ given in (A27) and the reflection factors r_λ given in (A18).

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